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On Kolmogorov's inertial-range theories

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Consistency and uniqueness questions raised by both the 1941 and 1962 Kolmogorov inertial-range theories are examined. The 1941 theory, although unlikely from the viewpoint of vortex-stretching physics, is not ruled out just because the dissipation fluctuates; but self-consistency requires that dissipation fluctuations be confined to dissipation-range scales by a spacewise mixing mechanism. The basic idea of the 1962 theory is a self-similar cascade mechanism which produces systematically increasing intermittency with a decrease of scale size. This concept in itself requires neither the third Kolmogorov hypothesis (log-normality of locally averaged dissipation) nor the first hypothesis (universality of small-scale statistics as functions of scale-size ratios and locally averaged dissipation). It does not even imply that the inertial range exhibits power laws. A central role for dissipation seems arbitrary since conservation alone yields no simple relation between the local dissipation rate and the corresponding proper inertial-range quantity: the local rate of energy transfer. A model rate equation for the evolution of probability densities is used to illustrate that even scalar nonlinear cascade processes need not yield asymptotic log-normality. The approximate experimental support for Kolmogorov's hypothesis takes on added significance in view of the wide variety of *a priori* admissible alternatives.

If the Kolmogorov law $E(k) \propto k^{-\frac{5}{3}-\mu}$ is asymptotically valid, it is argued that the value of μ depends on the details of the nonlinear interaction embodied in the Navier–Stokes equation and cannot be deduced from overall symmetries, invariances and dimensionality. A dynamical equation is exhibited which has the same essential invariances, symmetries, dimensionality and equilibrium statistical ensembles as the Navier–Stokes equation but which has radically different inertial-range behaviour.

1. Introduction

Kolmogorov's (1941) similarity theory introduced the hypothesis of a universal, isotropic, homogeneous statistical distribution for the small scales of motion in high-Reynolds-number incompressible turbulence. The theory envisaged a cascade of kinetic energy from large scales (low wavenumbers) to small scales (large wavenumbers) which was local in scale size, or wavenumber, and in which all statistical information about the large scales was lost, save for the mean energy-cascade rate itself. Kolmogorov formulated the theory as two specific hypotheses. First, that the n-variate distributions of the velocity differences $\mathbf{u}(\mathbf{x}+\mathbf{r}, t) - \mathbf{u}(\mathbf{x}, t)$ are universal isotropic functions solely of the difference vectors $\mathbf{r}$, the kinematic viscosity ν and the mean rate of energy dissipation per unit mass ϵ, provided that all the vectors $\mathbf{r}$ are small compared with macroscales of the turbulence. Second, that when, in addition, the vectors $\mathbf{r}$ are large compared with dissipation-range scales the distributions are independent of ν.

The two hypotheses lead immediately, by dimensional analysis, to explicit functional forms for moments of velocity differences in the inertial, or ν-independent, range. In particular,

$$\langle |\mathbf{u}(\mathbf{x}+\mathbf{r},t)-\mathbf{u}(\mathbf{x},t)|^n\rangle = B_n(\epsilon r)^{\frac{1}{3}n}, \tag{1.1}$$

where $\langle\ \rangle$ denotes an average over an appropriate ensemble and the B_n are universal constants. It follows also that the characteristic wavenumber which marks the transition from the inertial to dissipation range is $k_d = (\epsilon/\nu^3)^{\frac{1}{4}}$. On the assumption that the spectrum in the dissipation range falls off rapidly enough with increasing wavenumber, (1.1), with $n = 2$, leads by Fourier transformation to the inertial-range spectrum law

$$E(k) = C\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}, \tag{1.2}$$

where C is, again, a universal constant and the energy spectrum is defined so that the velocity variance is

$$2\int_0^\infty E(k)\,dk.$$

Kolmogorov's 1941 theory has achieved an embarrassment of success. The '$-\frac{5}{3}$'-spectrum has been found not only where it reasonably could be expected but also at Reynolds numbers too small for a distinct inertial range to exist and in boundary layers and shear flows where there are substantial departures from isotropy, and such strong effects from the mean shearing motion that the stepwise cascade appealed to by Kolmogorov is dubious. Measurements at high Reynolds numbers not only support the universality of C over 1000-fold variations in ϵ, but also are consistent with the prediction, from Kolmogorov's first hypothesis, that the dissipation-range spectrum scales according to

$$E(k) = C\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}f(k/k_d), \tag{1.3}$$

where $f(x)$ is a universal function satisfying $f(0) = 1$ (Grant, Stewart & Moilliet 1962).

In contrast to the corroboration of (1.2) and (1.3), experiments during the past 10 years do not support the predictions of (1.1) for higher order moments. According to the latter, the normalized moments

$$\langle |\mathbf{u}(\mathbf{x}+\mathbf{r},t)-\mathbf{u}(\mathbf{x},t)|^n\rangle / \langle |\mathbf{u}(\mathbf{x}+\mathbf{r},t)-\mathbf{u}(\mathbf{x},t)|^2\rangle^{\frac{1}{2}n}$$

are universal numbers, independent of ϵ and r, if r is in the inertial range. Instead, the measurements yield values which increase dramatically with n and $1/r$ at high Reynolds numbers, indicating strong scale-dependent intermittency at small scales. Moreover, at fixed ϵ and r, the intermittency at a scale r in the inertial range appears to increase with L/r, where L is a macroscale characteristic of the energy-containing range. The experiments are inconclusive, because it is uncertain whether the Reynolds numbers are large enough to produce an asymptotic regime, particularly with respect to higher statistics. Nevertheless, the data suggest that all is not well with the 1941 theory.

Some of the early evidence on small-scale intermittency, together with further thoughts about the nature of the turbulent energy cascade, led Kolmogorov (1962) and Oboukhov (1962) to modify the 1941 theory so as to assign a key role to the statistics of the spatial distribution of dissipation.

The dissipation per unit mass at $(\mathbf{x}, t)$ is

$$\tilde{\epsilon}(\mathbf{x}, t) = \nu(\partial u_i/\partial x_j)\,[\partial u_i/\partial x_j + \partial u_j/\partial x_i], \tag{1.4}$$

and a local spatial average may be defined by

$$\tilde{\epsilon}_l(\mathbf{x}, t) = \int_{|\mathbf{y}|<t} \tilde{\epsilon}(\mathbf{x}+\mathbf{y}, t)\, d^3\mathbf{y}. \tag{1.5}$$

Kolmogorov (1962) replaced the first hypothesis of the 1941 theory by a modified hypothesis, which we shall state as follows. Suppose that $\mathbf{x}$ and $\mathbf{x}+\mathbf{r}$ both lie within a region of size l and that $l \ll L$, where L is the macroscale. Then the n-variate distributions of $\mathbf{u}(\mathbf{x}+\mathbf{r}, t) - \mathbf{u}(\mathbf{x}, t)$, for a given value of $\tilde{\epsilon}_l(\mathbf{x}, t)$, are universal functions of only ν, $\tilde{\epsilon}_l(\mathbf{x}, t)$, l and the arguments $\mathbf{r}$. The second hypothesis of the 1962 theory is, as before, that dependence on ν disappears if r is large compared with dissipation-range scales. In addition, Kolmogorov now added a third hypothesis, which makes a highly specific assumption about the statistics of $\tilde{\epsilon}_l(\mathbf{x}, t)$. He supposed that $\tilde{\epsilon}_l(\mathbf{x}, t)$ is log-normal for $l \ll L$, with a variance given, in the statistically homogeneous case, by

$$\sigma_l^2 = A + \mu \ln(L/l). \tag{1.6}$$

Here $\sigma_l^2 \equiv \langle [\ln(\tilde{\epsilon}_l/\epsilon)]^2 \rangle$, μ is a universal constant and A is a constant which depends on macroscale statistics. We may note that, by (1.5) and homogeneity, $\epsilon = \langle \tilde{\epsilon}(\mathbf{x}) \rangle = \langle \tilde{\epsilon}_l(\mathbf{x}) \rangle$, and the averages depend on neither $\mathbf{x}$ nor l.†

By dimensional analysis, the first two hypotheses yield

$$\langle |\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})|^n \rangle = B_n \langle [\tilde{\epsilon}_r(\mathbf{x})]^{\frac{1}{3}n} \rangle\, r^{\frac{1}{3}n}, \tag{1.7}$$

where we suppress time arguments. The B_n are universal numbers. According to the third hypothesis, (1.7) has the explicit form

$$\langle |\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})|^n \rangle = B'_n (\epsilon r)^{\frac{1}{3}n} (r/L)^{\frac{1}{18}\mu n(3-n)} \tag{1.8}$$

(cf. Gurvich & Yaglom 1967). Equation (1.2) is thereby replaced by

$$E(k) = C\epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} (kL)^{-\frac{1}{9}\mu}, \tag{1.9}$$

and the normalized inertial-range velocity-difference moments are

$$\langle |\Delta\mathbf{u}(\mathbf{x}, \mathbf{r})|^n \rangle / \langle |\Delta\mathbf{u}(\mathbf{x}, \mathbf{r})|^2 \rangle^{\frac{1}{2}n} = B'_n (B'_2)^{-\frac{1}{2}n} (L/r)^{\frac{1}{18}\mu n(n-2)}, \tag{1.10}$$

where $\Delta\mathbf{u}(\mathbf{x}, \mathbf{r}) = \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})$.

The dependence on n and on L/r in (1.10) is at least roughly consistent with the experimental evidence on higher moments. Both moment measurements and determinations of the spectrum of $\tilde{\epsilon}(\mathbf{x})$, which we shall discuss later, suggest

† Kolmogorov (1962) stated his theory more generally, for inhomogeneous turbulence, but the homogeneous case will be sufficient for our purposes.

that $\mu \sim 0{\cdot}5$. With this value of μ, (1.9) is not distinguishable experimentally from the original $-\frac{5}{3}$-law. Further support for the 1962 theory is provided by measurements of velocity derivatives $\partial \mathbf{u}/\partial \mathbf{x}$, which yield some fairly direct evidence that the probability distribution of $\tilde{\epsilon}(\mathbf{x})$ is approximately log-normal. Some recent symposium papers which discuss the experimental results are Monin (1967), Gibson, Stegun & Williams (1970), Stewart, Wilson & Burling (1970), Dutton & Deaven (1972), Gibson & Masiello (1972), Van Atta & Park (1972) and Wyngaard & Pao (1972). Other significant papers on the small-scale structure of laboratory and geophysical flows include Batchelor & Townsend (1949), Grant & Moilliet (1962), Gurvich & Zublowskii (1963), Pond, Stewart & Burling (1963), Pond & Stewart (1965), Gibson, Stegun & McConnell (1970), Wyngaard & Tennekes (1970), Frenkiel & Klebanoff (1971), Kuo & Corrsin (1971), Sheih, Tennekes & Lumley (1971), Kuo & Corrsin (1972) and Tennekes & Wyngaard (1972). These references provide fairly exhaustive citations of other work.

The present paper has two principal motivations. The first is to argue that the 1941 theory is by no means logically disqualified merely because the dissipation rate fluctuates. On the contrary, we find that at the level of crude dimensional analysis and eddy-mitosis pictures (which is the extent of the contact usually made with the Navier–Stokes equation in briefs for either theory) the 1941 theory is as sound a candidate as the 1962 theory. This does not imply that we espouse the 1941 theory. On the contrary, the theory is made implausible by the basic physics of vortex stretching. The point is that this question cannot be decided *a priori*; some kind of *non-trivial* use must be made of the Navier–Stokes equation.

Our second principal purpose is to point out, with the aid of models, that, once the 1941 theory is abandoned, a Pandora's box of possibilities is opened. The 1962 theory of Kolmogorov seems arbitrary, from an *a priori* viewpoint, not only because the third hypothesis is so specific but because there are logical alternatives to the first hypothesis as well. Several authors have questioned whether the idea of a multistage energy cascade, which underlies both Kolmogorov theories, is in fact valid (Townsend 1951; Corrsin 1962; Saffman 1968; Tennekes 1968). We make the point, instead, that even in the general framework of some kind of self-similar cascade, and of intermittency which increases with the number of cascade steps, the 1962 theory is only one of many possibilities. From this position, the available experimental data take on added significance. The Reynolds numbers are not high enough to yield results which could conclusively verify any given theory of the small scales, nor are significantly higher Reynolds numbers foreseeable. Nevertheless, what data there are do give substantial support to the 1962 Kolmogorov theory, and closely related alternatives, over other possibilities which must be logically admitted. This, it is hoped, offers valuable clues to better understanding of the physics of turbulence and to analytical attacks.

How a theoretical attack on the inertial-range problem should proceed is far from clear. No *formal* analysis by means of perturbation theory, the moment-equation hierarchy, or renormalization techniques can settle whether (1.9) is a valid equation and, if so, whether $\mu = 0$ or not. Orszag (1966) has shown that

$\mu = 0$ is formally consistent with every order of the moment hierarchy, and indeed arises from the hierarchy under the assumption that the moments and the cumulants of any given order go as the same power of k. But if this restriction is relaxed, then $\mu \neq 0$ also is formally consistent. The Eulerian renormalized perturbation series has the formal solution $E(k) \propto k^{-\frac{3}{2}}$ at every order, while low-order closure schemes that are invariant to random Galilean transformations lead naturally to $-\frac{5}{3}$ (Kraichnan 1964). Such schemes cannot embody the higher statistics associated with intermittency build-up. The real question concerns which, if any, of the formal solutions imply probability distributions that satisfy all realizability inequalities. The present paper does not face these questions. Instead, we point out some of the varied possibilities that arise from the basic concepts of Kolmogorov's theories and hope, thereby, to help define what the problems are.

2. Is the 1941 theory inconsistent?

Kolmogorov traces the origins of the 1962 theory to a remark by Landau (cf. Landau & Lifshitz 1959) which questioned the universality of C in (1.2). Landau's point, in essence, is that C is not invariant to the composition of subensembles because the left-hand side of (1.2) is an average while the right-hand side is the $\frac{2}{3}$ power of an average. Since the magnitude of $\tilde{\epsilon}(\mathbf{x}, t)$, averaged over the spatial domain of a flow, depends on the macroscale L and the intensity of large-scale excitation, it follows that C cannot be universal if (1.2) is asserted for flows with arbitrary statistical distributions of large-scale parameters. This difficulty arises whether the averages are taken as spatial averages over a super-large flow containing different macroregimes, or as corresponding ensemble averages over realizations of a single macroregime.

However, the sensitivity of (1.2) to macrostatistics is really not to the point. The 1941 theory is intended to describe a universal statistical state attained by small scales, and therefore it should be applied only to subregions of a flow sufficiently small compared with gross dimensions that cascading has been able to set up that state. Statistical ensembles appropriate to the 1941 theory should have such subregions as typical realizations, or should describe statistically homogeneous flows that already display the universal statistical state in the macroscales. The real question to be answered is whether the putative universal state is self-consistent and is a consequence of the cascade process.

Landau's objection can also be raised for subregions of flow regions already small enough for the universal state to be established. The only way in which a universal value of C can survive is if the statistical fluctuations of $\tilde{\epsilon}(\mathbf{x}, t)$ are confined to dissipation-range scales. That is, if

$$\tilde{\epsilon}_l(\mathbf{x}, t) \approx \epsilon, \tag{2.1}$$

for scales l which are well within the inertial range. This does *not* imply that there are no substantial fluctuations in energy *transfer* on the scale l. It is essential to make the logical distinction between energy transfer and energy dissipation, and we shall now digress to set up a formalism for doing this.

Consider flow in an infinite domain, or a cyclic box, and make the decomposition

$$\mathbf{u}(\mathbf{x},t) = \sum_{n=0}^{\infty} \mathbf{u}^n(\mathbf{x},t), \tag{2.2}$$

where $\mathbf{u}^n(\mathbf{x},t)$ is the total contribution from all wavenumbers in the band $2^{n-1} < k/k_0 < 2^n$, except that $\mathbf{u}^0(\mathbf{x},t)$ represents the band $0 < k/k_0 < 1$. Here k_0 is any characteristic wavenumber of the macroscale motion. The total energy (divided by density) is

$$\int |\mathbf{u}(\mathbf{x},t)|^2 d^3\mathbf{x} = \sum_{n=0}^{\infty} \int |\mathbf{u}^n(\mathbf{x},t)|^2 d^3\mathbf{x} \tag{2.3}$$

and the total dissipation is

$$\int \tilde{\epsilon}(\mathbf{x},t)\, d^3\mathbf{x} = \sum_{n=0}^{\infty} \int \tilde{\epsilon}^n(\mathbf{x},t)\, d^3\mathbf{x}, \tag{2.4}$$

where $\tilde{\epsilon}^n(\mathbf{x},t)$ is given by (1.4), with u^n in place of u. The diagonal form of (2.3) and (2.4) makes it convenient to identify the $\mathbf{u}^n(\mathbf{x},t)$ with the various scales of motion, rather than the velocity differences $\mathbf{u}(\mathbf{x}+\mathbf{r},t)-\mathbf{u}(\mathbf{x},t)$ usually used. The same kind of decomposition can be made for general (non-rectangular) boundary conditions by using an appropriate set of orthogonal eigenfunctions.

The energy density associated with band n at a particular point $\mathbf{x}$ is not a well-defined concept, because there are cross-terms. Nevertheless, the global relations (2.3) and (2.4) make it illuminating to deal with the proper-energy density $\frac{1}{2}|\mathbf{u}^n(\mathbf{x},t)|^2$ and proper-dissipation rate $\tilde{\epsilon}^n(\mathbf{x},t)$ even though they do not sum locally to the actual energy density and dissipation rate. The equation of motion for $\mathbf{u}^n(\mathbf{x},t)$ is

$$(\partial/\partial t - \nu\nabla^2)\, u_i^n(\mathbf{x},t) = -P_{ij}^n(\nabla)\,[u_s(\mathbf{x},t)\, \partial u_j(\mathbf{x},t)/\partial x_s], \tag{2.5}$$

where $P_{ij}^n(\nabla)$ is a band-limited solenoidal projection operator, defined in the Fourier representation by

$$P_{ij}^n(\mathbf{k}) = \begin{cases} \delta_{ij} - k_i k_j/k^2 & (k \text{ in } n\text{th band}), \\ 0 & (k \text{ not in } n\text{th band}). \end{cases}$$

It follows that the rate of transfer of proper-energy density into all bands above n is

$$\tilde{\Pi}^n(\mathbf{x},t) = -\sum_{m=n+1}^{\infty} u_i^m P_{ij}^m(\nabla)\,[u_s \partial u_j/\partial x_s], \tag{2.6}$$

while the mean rate of energy transfer from lower bands to all bands above n is

$$\Pi^n(t) = \langle \tilde{\Pi}^n(\mathbf{x},t) \rangle. \tag{2.7}$$

In a statistically steady state, conservation implies that $\Pi^n(t) = \epsilon$, for n in the inertial range. But this does *not* mean that $\tilde{\Pi}^n(\mathbf{x},t) = \tilde{\epsilon}(\mathbf{x},t)$ in the statistically steady state. Both $\mathbf{u}^n(\mathbf{x},t)$ and, consequently, $\tilde{\Pi}^n(\mathbf{x},t)$ exhibit fluctuations as a result of transient imbalances in the cascade and of spatial transport of energy within the bands. Now we must examine the consequences of the case in which the fluctuations in $\tilde{\Pi}^n(\mathbf{x},t)$ are on a larger spatial scale than those of

$\tilde{\Pi}^{n'}(\mathbf{x},t)$, if $n' \gg n$, and, especially, are on a larger spatial scale than those of $\tilde{\epsilon}(\mathbf{x},t)$. For this to be dynamically consistent, we must suppose that there is a strong mixing of energy *in space*, as well as the cascade in wavenumber. The characteristic time for spacewise diffusion of energy on the scale of band n must not be larger in order of magnitude than that for transfer of energy to higher bands. A further implication is that $\mathbf{u}^n(\mathbf{x},t)$ and $\mathbf{u}^{n'}(\mathbf{x},t)$ are statistically independent if $n' \gg n$.

The equations of motion give some support, in a general way, to the possibility that there exists a spacewise diffusion mechanism adequate to validate the 1941 theory. Let us truncate (2.5) by retaining on the right-hand side only the contributions $u_s^n(\mathbf{x},t)$ and $u_j^n(\mathbf{x},t)$ to the u fields. In the absence of viscosity, the truncated system is conservative and satisfies Liouville's theorem (Lee 1952). Left to itself, this system plausibly should relax to an equilibrium distribution in which any initial strong intermittencies in $\mathbf{u}^n(\mathbf{x},t)$ are relaxed and $\mathbf{u}^n(\mathbf{x},t)$ is multivariate normal. It seems clear on dimensional grounds that the characteristic time for this relaxation is $\sim (k_n v_n)^{-1}$, where k_n and v_n are the characteristic wavenumber and amplitude level for $\mathbf{u}^n(\mathbf{x},t)$. This is also the magnitude of the time constant for yielding up of the energy in $\mathbf{u}^n(\mathbf{x},t)$ to higher bands, according to the Kolmogorov cascade ideas for the full, non-truncated system.†

Returning to the actual Navier–Stokes equation, we probably must concede that there exist spacewise diffusion effects of sufficient strength to make the 1941 theory *a priori* possible. Whether these effects, which constitute cross-linkings of spatially local cascade chains, actually do suppress the build-up of intermittency appealed to in the 1962 theory is another question. We have two reasons for feeling (but not proving!) that they do not. One is the results of the cascade models presented later in this paper, which exhibit a build-up of intermittency with the number of cascade steps, despite a cross-linking mechanism. The other lies in some simple physics of the vortex-stretching process.

In closing this section, we want to make the point that $\tilde{\epsilon}_l(\mathbf{x},t)$ is not an inertial-range quantity when l is an inertial-range scale. Instead it is the integral of a dissipation-range quantity. The proper inertial-range quantity is the energy transfer rate, as given by $\tilde{\Pi}^n(\mathbf{x},t)$ or some similar construction. We have shown that the consistency of the 1941 theory requires that $\tilde{\epsilon}(\mathbf{x},t)$ fluctuations be essentially confined to scales of order k_d^{-1}. There would appear to be no logical basis, on the 1941 theory, for supposing that the low-k tail of the spectrum of $\tilde{\epsilon}(\mathbf{x},t)$ obeys inertial-range scaling, which would imply that $F_\epsilon \sim \epsilon^2 k^{-1}$ for inertial-range k. On the contrary, if the spectrum has the form $F_\epsilon \sim \epsilon^2(k/k_d)^s k^{-1}$ for inertial-range k, we have no way, on the basis of simple qualitative considerations, of deciding what s should be.

† It is of interest, at this point, that we can exhibit explicitly a self-consistent dynamical system which embodies the local-in-wavenumber energy cascade invoked by the 1941 theory and yields both the $\frac{5}{3}$-spectrum and an $\tilde{\epsilon}(\mathbf{x},t)$ which fluctuates only on dissipation-range scales. This is the direct-interaction approximation (realizable by a model construction) for a modified Navier–Stokes equation in which the convection effects of given scales on much smaller scales are consistently removed (Kraichnan 1964).

3. Vortex stretching and intermittency

The terms 'scale of motion' and 'eddy of size l' appear repeatedly in treatments of the inertial range. One gets an impression of little, randomly structured and distributed whirls in the fluid, with the cascade process consisting of the fission of the whirls into smaller ones, after the fashion of Richardson's poem. This picture seems to be drastically in conflict with what can be inferred about the qualitative structure of high-Reynolds-number turbulence from laboratory visualization techniques and from plausible application of the Kelvin circulation theorem.

Batchelor (1952) argued that a random flow should, on the average, separate irreversibly the vertices of an initially compact small volume moving with the fluid. Incompressibility then implies that the surfaces of the volume must be drawn towards each other, producing an extended thin structure out of the initially compact blob. An eventual ribbon-like structure seems most likely since it is improbable that the net stretching in any two directions should be nearly equal. Recently Cocke (1969) and Orszag (1970) have proved that a line element, or a surface element moving with an isotropically turbulent fluid is, in fact, drawn out on the average.

The stretching mechanism has led a number of authors to conjecture that the small-scale structure should consist typically of extensive thin sheets and ribbons of vorticity, drawn out by the straining action of their own shear fields (e.g. Townsend 1951; Batchelor 1953; Kraichnan 1959; Corrsin 1962; Saffman 1968; Tennekes 1968). In this picture, the randomness lies in the distribution of thickness and extension of the thin sheets and ribbons, and in the way they are folded and tangled through the fluid. A typical small-scale structure is thought to be small in one or two dimensions only, not in the third.

It is difficult to test the picture above by laboratory measurements. However, Kuo & Corrsin (1972) have recently tried, and they do infer, from two-point hot-wire measurements, that the intermittent small-scale structures observed at high Reynolds numbers are typically ribbons or tubes of activity, rather than compact blobs. An elementary supporting visualization consists of stirring ink into a vessel of water. Of course the resulting sheets and ribbons of ink are not vorticity, but observation of this phenomenon makes similar behaviour for the vorticity plausible.

Now consider a specific initial-value problem. Let the initial flow of very high Reynolds number consist of a random array of ring vortices, all with the same ring diameter, tube cross-section and vortex strength. The rings may or may not be linked, but they do not intersect. Assume that viscous diffusion is negligible until the rings are stretched to very much smaller cross-sections than they have initially. On the basis of the preceding discussion, we expect that the collective shear field will typically stretch the rings into long, tangled, but still non-intersecting ribbons. Since the vorticity does not diffuse, the volume of fluid containing vorticity stays the same during the stretching, and in this sense there is no increase in intermittency. However, Kelvin's circulation theorem implies that the total vortex strength per unit length of ribbon stays constant during the stretching, so that the local vorticity amplitude in the ribbon is inversely proportional to the local ribbon cross-section. Since the stretching is stochastic, an initial δ-function distribution of vorticity amplitude in the rings develops a statistical spread. If there is effective statistical independence of successive stretchings, then each stretched ribbon element of given vorticity amplitude develops a further statistical spread. The result is that the vorticity-amplitude distribution displays ever-increasing intermittency in space, until the ribbons are stretched to a thinness such that viscosity can no longer be neglected.

The simplest mathematical model of the increase of intermittency of vorticity amplitude during the assumed random stretching is that of Saffman (1970). He assumes that

$$d\omega/dt = b(t)\,\omega, \tag{3.1}$$

where ω is the vorticity in a fluid element and $b(t)$ is the effective stretching. This gives

$$\ln\left[\omega(t)/\omega(0)\right] = \int_0^t b(s)\,ds, \tag{3.2}$$

so that $\omega(t)$ becomes asymptotically log-normal for t long compared with characteristic correlation times of $b(s)$. We shall see in the next section that the asymptotic distribution need not be, and probably is not, actually log-normal. However, it seems inevitable that the vortex-stretching picture does lead to some kind of increasingly intermittent vorticity distribution. To avoid this conclusion, one must appeal to some kind of negative correlation between the degree of past stretching and probability of future stretching; that is, a tendency towards alignment of strongly stretched ribbons in the ambient shear in such a way that future stretching is minimized. This cannot be ruled out *a priori*, since vorticity and shearing are functionally related, but it seems an awkward assumption.

The vortex-stretching picture, with its implied increase of intermittency during cascade, seems quite foreign to the 1941 theory. The spacewise mixing within given scale sizes, which we found essential to that theory, would be associated, in the vortex-stretching picture, with the spatial convection and bending, or bellying, of the ribbons. It would then be a concomitant rather than a competitor of the intermittency-increasing stretching. In the preceding section we invoked a tendency towards absolute statistical equilibrium within the bands $\mathbf{u}^n(\mathbf{x}, t)$ as giving possible support to the 1941 theory. The vortex-stretching picture suggests that energy and vorticity are squeezed into higher bands, at each cascade step, before this tendency can be effective.

The vortex-stretching picture gives general qualitative support to the 1962 Kolmogorov theory, provided that the stretching is associated with successive instabilities of the vortex sheets, corresponding to the random cascade steps of that theory. Several authors, however, have questioned the whole concept of a hierarchy of random cascade steps. Corrsin (1962) and Saffman (1968, 1970) propose models in which the small-scale structure consists of shear layers whose thickness is the Kolmogorov dissipation scale, but which are coherent over much larger distances: the macroscale in Corrsin's case and the Taylor microscale in Saffman's. Measured on inertial-range scales, these structures are effectively shear discontinuities, the low wavenumber tail of whose spectral decomposition represents the inertial-range spectrum. Chorin (1970, unpublished manuscript) proposes a variant in which the inertial range is the spectral tail of concentrated line vortices.

A surface shear discontinuity gives a low-k tail of the form $E(k) \propto k^{-2}$, while a line vortex has a tail like $E(k) \propto k^0$. Thus, in order to get something close to $E(k) \propto k^{-\frac{5}{3}}$, in accord with experiment, a cusp-like velocity distribution would be needed at the discontinuity in the case of surface discontinuities, while an extended vorticity distribution, away from the centre-line, would be needed in the case of line vortices.

A generally unappreciated point is that the inertial-range energy cascade is local in wavenumber, even when the inertial-range spectrum is the spectral tail of coherent discontinuous structures, like those suggested by Corrsin, Saffman and Tennekes. We shall illustrate this with Burgers' equation. Here it is known that a freely decaying, infinite-Reynolds-number flow with cyclic boundary conditions degenerates into a velocity distribution consisting of a single decaying sawtooth shock wave in each cyclic cell, yielding an energy spectrum $E(k) \propto k^{-2}$. Let the cyclic cell size equal 2, with the sloping sawtooth edge centred on $x = 0$ in the primary cell. The velocity field in this cell is then

$$u(x,t) = x/(t-t_0) \quad (|x| < 1),$$

where t_0 is a virtual time origin. The Fourier decomposition

$$u(x,t) = \sum_{n=-\infty}^{\infty} u_n(t) \exp(in\pi x)$$

then gives

$$u_n(t) = 0 \quad (n \text{ even}), \quad u_n(t) = i[\pi n(t-t_0)]^{-1} \quad (n \text{ odd}), \tag{3.3}$$

while the transform of the inviscid Burgers equation is

$$du_n/dt = -\tfrac{1}{2} in \sum_{m=-\infty}^{\infty} u_{n-m} u_m. \tag{3.4}$$

The point now is that the contributions to the right-hand side of (3.4) are local in n space, dominated by $|n-m| \sim |n|$ and $|m| \sim |n|$. It is easily verified that (for n odd) the total contribution to (3.4) from all $|m| \leqslant m_1 < |n|$ is

$$\frac{1}{\pi} in \sum_{m=1}^{m_1} (n^2-m^2)^{-1}.$$

For $m_1 \ll n$, this sum is $\sim im_1/n$ and is in phase with u_n, representing an energy input to u_n which vanishes for $m_1/n \to 0$. Similarly, the contribution from all $|m| \geqslant m_1 > |n|$ is

$$\frac{1}{\pi} in \sum_{m=m_1}^{\infty} (n^2-m^2)^{-1},$$

which is out of phase with u_n, representing an energy drain to higher wavenumbers. Again, this contribution is local in the sense that it vanishes for $n/m_1 \to 0$. The principle which this example illustrates is that localness of energy transfer in Fourier space depends on the exponent of the spectrum power law, and not on coherence properties.

At present there is not enough solid knowledge about the Navier–Stokes equation to say with assurance that Kolmogorov's idea of multiple random cascade steps is qualitatively correct. We feel, however, that the models of

Corrsin, Tennekes and Saffman are unlikely candidates, for two reasons. The first is the difficulty of producing an inertial-range spectrum close to experiments, which fit well the $-\frac{5}{3}$-law. There seem to be two ways of escaping the k^{-2} law associated with an extensive coherent shear-discontinuity surface. Either one appeals to some cusp-like behaviour at the discontinuity, for which no physical basis has been offered, or else one supposes that the surface of discontinuity is crinkled on inertial-range scales, so as to give destructive interference at low wavenumbers. The second possibility would appear to give a model which differs little from Kolmogorov's, for one must have some mechanism to make the crinkling. Our second source of discontent is more basic, but also more vague. It seems simply implausible that, at very high Reynolds numbers, the shear layers should shrink down to a dissipation-scale thickness as a result of only one or two instability breakdowns. What is there that could lock out myriad instabilities of many kinds in the high-Reynolds-number fluid? The situation with Burgers' equation is quite different; there are no instabilities of the freely decaying velocity field.

Even if the underlying idea of a multistage breakdown is correct, Kolmogorov's first and third hypotheses of 1962 do not necessarily follow. In the next section, we shall discuss the third hypothesis, which is the easier to take up first because it is so specific.

4. Log-normality and alternatives

It is not clear whether Kolmogorov (1962) meant the log-normal law to be literally true, or simply representative of distributions in which intermittency increased systematically with decreasing scale size. The companion paper of Oboukhov (1962) suggests the latter view. Whatever the original intention, log-normality has been taken very literally by experimenters seeking to interpret their data.

We shall now present two reasons why it is doubtful that the small scales exhibit asymptotic distributions that are really log-normal. The first has to do with the nonlinearity of the dynamical processes; the second is connected with the fact that the sum of independent log-normal variables is not log-normal. We shall point out also that, even if the asymptotic distributions were log-normal, moment ratios in the non-asymptotic region (which is the region experiments are limited to) could differ drastically from log-normal values.

Equation (3.1) typifies dynamical equations which lead asymptotically to log-normality. It is linear, the coefficient process $b(t)$ is independent of $\omega(t)$, and it involves a single dependent variable. The precise conditions on $b(t)$ which make

$$\int_0^t b(s)\,ds$$

asymptotically normal are discussed by Lumley (1972) and Rosenblatt (1972). Clearly, if $b(t)$ happens to be normal, then $\omega(t)$ is log-normal for all t, if $\omega(0)$ is statistically sharp. Note that the composition law for normal variables implies that the product of two log-normal variables is log-normal.

An example of a flow problem which leads unambiguously to (3.1) and log-normal statistics is the elongation of infinitesimal line elements that move with an incompressible fluid describing random, statistically homogeneous and isotropic motion (Cocke 1969; Orszag 1970). The rate of elongation of any given line element of length $\omega(t)$ clearly is proportional to $\omega(t)$, yielding (3.1), with statistics for $b(t)$ which are a functional of the flow statistics. Now, let the fluid start from rest at $t = 0$, describe a statistically homogeneous and isotropic motion and come to rest at $t = t_1$. Then let it describe a similar but statistically independent motion for $t_1 < t < t_2$, and so forth, where the t_n are any set of roughly equally spaced times.

The statistical isotropy, homogeneity and independence conditions assure that $b(t)$ for $t > t_n$ is statistically independent of $\omega(t)$, and $b(t)$ for $t < t_n$, there being no possible correlation due to orientation of the line element. That is, the statistics of future elongation depend solely on the scalar length $\omega(t)$ at each time t_n. It then follows from the central limit theorem that

$$\int_0^t b(s)\,ds$$

is asymptotically normal as $t \to \infty$. Cocke (1969) shows that the mean logarithmic elongations

$$\left\langle \int_{t_n}^{t_{n+1}} b(s)\,ds \right\rangle$$

are positive.†

Now consider (3.1) again as a model for vorticity increase due to stretching of vortex lines in the shear field. The shear and vorticity fields are intimately related, so that it is no longer really plausible to take $b(t)$ as statistically independent of $\omega(t)$ in (3.1). A more plausible model would be

$$d\omega/dt = a(t)\,\omega^2 - \eta\omega^3, \tag{4.1}$$

where $\omega > 0$, $\eta > 0$ and $a(t)$ is a random process. This equation may be considered a crude simulation of the increase of vorticity by a cascade process like that called for by the 1962 Kolmogorov theory. The local vorticity level ω increases stochastically, but at a rate which, on the average, is proportional to ω itself. The $\eta\omega^3$ term is an extremely crude model of the viscous damping: the damping factor $\eta\omega^2$ would be proportional to νk^2 if, as is qualitatively plausible from a picture of vorticity increase by stretching, the characteristic wavenumbers for the vorticity field rise as ω does. We take $a(t)$ as a stationary process, although it would be more realistic to let the characteristic frequency of $a(t)$ increase with ω.

Now suppose that $\omega(0) \ll 1$, $\eta \ll 1$ and $\langle [a(t)]^2 \rangle \sim 1$. The damping term can be neglected in any realization until $\omega(t) \gg 1$ occurs in that realization. Prior to that time, (4.1) has the approximate solution

$$[\omega(t)]^{-1} - [\omega(0)]^{-1} = -\int_0^t a(s)\,ds. \tag{4.2}$$

† In a later paper, Cocke (1971) finds that the mean logarithmic *rate* of elongation tends to zero as $t \to \infty$. Our construction here of a flow which is statistically independent in successive finite time intervals is a counter example, suggesting that this finding rests on an improper application of a central limit theorem for dependent variables.

Normality of the right-hand side, by virtue of the central limit theorem, as $t \to \infty$ now implies that $1/\omega(t)$, rather than $\ln[\omega(t)]$, has an asymptotically normal distribution, if $\omega(0)$ is statistically sharp. If the damping correction is taken into account, the distribution departs from $(1/\omega)$-normality at large enough ω values.

The importance of this example lies not in the specific model, which we do not propose seriously, but in the demonstration that nonlinearity can invalidate simple arguments for log-normality. Great caution is required in making central limit arguments. Thus (4.2) can be rewritten as

$$\ln[\omega(t)/\omega(0)] = \int_0^t a(s)\,\omega(s)\,ds. \tag{4.3}$$

But now the central limit theorem cannot be legitimately applied to the right-hand side because $\omega(s)$ never becomes independent of early values. A further example in which nonlinearity is associated with departure from log-normality is provided by the model of inertial-range cascade presented in § 5, to follow.

Yaglom (1966), Gurvich & Yaglom (1967), Novikov (1971) and others have considered models of the inertial-range cascade in which quantities associated with successive stages of the scale hierarchy are related according to

$$\phi_{n+1} = a_{n+1}\phi_n, \tag{4.4}$$

where the a_n are independent positive random variables. One choice for ϕ_n could be the absolute value of $\mathbf{u}^n(\mathbf{x}, t)$ at a given point. If $\phi_0 = 1$, it follows from the central limit theorem that ϕ_n has an asymptotically log-normal distribution as $n \to \infty$.

However, it is physically as reasonable to replace (4.4) with

$$\phi_{n+1,i} = \sum_{j=1}^{R} a_{n+1,ij}\phi_{n,j} \quad (i = 1, 2, \ldots, R), \tag{4.5}$$

where the $a_{n,ij}$ are random matrices, independent for different n. This would mean that the cascade step at each point would depend on the interrelation of several parameters, the R quantities $\phi_{n,i}$, instead of just one parameter. Perhaps the $\phi_{n,i}$ could be the shear at several nearby points, enough points to determine how stable or unstable the local velocity field is. But now, it no longer is true in general that $\phi_{n,i}$ has an asymptotically log-normal distribution, the reason being that addition of log-normal variables does not usually yield a log-normal sum.

If R is a small integer, it is to be expected that the departure of the asymptotic distribution from log-normality is weak, in the sense that the growth of high-order moments with n should not differ much from the way they would grow under (4.4). This is supported by the observation that adding together R statistically similar and independent variables reduces the normalized cumulant of order s by a factor $R^{\frac{1}{2}(2-s)}$, while multiplication by an independent factor a increases the normalized moment of order s by the factor $\langle a^s \rangle / \langle a^2 \rangle^{\frac{1}{2}s}$. If, say, a is exponentially distributed, or has any distribution with an exponential or Gaussian tail, then this second factor overwhelms the $R^{\frac{1}{2}(2-s)}$ factor at large s.

We wish also to call attention to a special situation which leads to exact log-normality. Consider

$$\mathbf{Y}_{n+1} = \mathbf{A}_{n+1}\mathbf{Y}_n \quad (\mathbf{Y}_0 = \mathbf{I}). \tag{4.6}$$

Here the $\mathbf{A}_n$ and $\mathbf{Y}_n$ are square matrices and $\mathbf{I}$ is the unit matrix. If the $\mathbf{A}_n$ are statistically independent matrices, the elements of $\mathbf{Y}_n$ are in general not asymptotically log-normal. However,

$$\det(\mathbf{Y}_n) = \det(\mathbf{A}_n)\det(\mathbf{A}_{n-1})\ldots\det(\mathbf{A}_1)$$

is asymptotically log-normal, and is log-normal for finite n if the $\det(\mathbf{A}_n)$ are all log-normal. Note that (4.6) is more restrictive than (4.5). In the latter, there are R dependent variables and R^2 coefficients, while in (4.6) the number of components of $\mathbf{A}_{n+1}$ equals the number of components of $\mathbf{Y}_n$.

It is clear from the preceding discussion that, in the absence of any present basis for discounting effects of nonlinearity and linear composition in the inertial-range cascade process, the hypothesis that $\tilde{\epsilon}(\mathbf{x},t)$ and $\tilde{\epsilon}_l(\mathbf{x},t)$ are asymptotically log-normal must be considered arbitrary. Moreover, there is *no* reason to suppose, except as an approximation, that these quantities should be log-normal at the finite Reynolds numbers and values of L/l which are observable.

However, it should be pointed out that the general non-survival of log-normality under addition does not mean that simultaneous log-normality of $\tilde{\epsilon}(\mathbf{x},t)$ and its space integral $\tilde{\epsilon}_l(\mathbf{x},t)$ is mathematically inconsistent, a question which has been raised by Mandelbrot (1972). It remains possible that the sum of suitably correlated log-normal variables is log-normal even though this is not true in general. Consider the system

$$\phi_0 = 1, \quad \phi_{1,i} = a_i\phi_0, \quad \phi_{2,ij} = a_{ij}\phi_{1,i}, \quad \phi_{3,ijm} = a_{ijm}\phi_{2,ij}, \tag{4.7}$$

where all indices $(i,j,\ldots)$ run from 1 to R and the positive coefficients a_i, a_{ij}, a_{ijm}, etc. are completely statistically independent except for the constraints

$$\sum_i a_i = 1, \quad \sum_j a_{ij} = 1, \quad \sum_m a_{ijm} = 1, \ldots . \tag{4.8}$$

Each variable $\phi_{n,ij\ldots}$ is a product of n statistically independent a-factors and therefore is log-normal as $n \to \infty$, while (4.8) implies that

$$\Sigma\,\phi_{m,ij\ldots} = \phi_{n,ij\ldots} \quad (m > n), \tag{4.9}$$

where the summation on the left is over all indices (after the comma) that do not appear on the right. The construction does not yield $\phi_{n,ij\ldots}$ which are log-normal for finite n because (4.7) does not permit log-normal univariate distributions for the positive variables a.

For comparison with experiments, the precise analytical form of the distribution is unimportant, both because asymptotic conditions cannot be expected in the relatively few cascade steps attainable at even the largest known Reynolds numbers and because there are severe experimental difficulties in measuring highly intermittent distributions (Tennekes & Wyngaard 1972). We wish, however, to stress a point previously made by Novikov (1971). Consider again the original simple scalar process (4.4), which leads asymptotically to log-normal ϕ_n. The normalized moments of ϕ_n are

$$\langle(\phi_n)^s\rangle/\langle(\phi_n)^2\rangle^{\frac{1}{2}s} = \prod_{m=1}^{n} \langle(a_m)^s\rangle/\langle(a_m)^2\rangle^{\frac{1}{2}s}.$$

These ratios depend on the specific statistics of the a_m, no matter how large n is. Unless the a_m happen themselves to be log-normal, the moment ratios for large n cannot be approximated, in general, by the ratios for a log-normal ϕ_n distribution.

A final point which perhaps needs recalling is that arguments for log-normality are completely inapplicable to individual wave-vector components of the velocity field of homogeneous turbulence in an infinite cyclic box. If statistical correlations are confined to finite distances, then the *univariate* distribution of every individual Fourier component is accurately normal, whatever the degree of intermittency of the $\mathbf{x}$-space velocity field (Kraichnan 1959; Lumley 1972). The band-limited fields $\mathbf{u}^n(\mathbf{x}, t)$, on the other hand, involve many Fourier components which are weakly statistically dependent (Kraichnan 1959), and they do express spatial intermittency.

5. A cascade model

We wish now to introduce a model of the inertial-range cascade which serves the dual purpose of illustrating that nonlinearity can effect departure from asymptotic log-normality and of providing a vehicle for exploring the impact, on intermittency growth, of the kind of spatial diffusion (cross-linking of cascade chains) appealed to in §2. The model posits an equation for the rate of change of the probability distribution of the $|\mathbf{u}^n(\mathbf{x}, t)|$ due to supposed sudden disintegrations of structures of one scale into ones of smaller scale. In its use of a rate equation, the model differs from those of Yaglom (1966), Gurvich & Yaglom (1967), Novikov & Stewart (1964) and Novikov (1971), which postulate relations of the general sort (4.4) between quantities describing an instantaneous statistically steady state. Our model is no less arbitrary than those cited.

The $\mathbf{u}^n(\mathbf{x}, t)$ were defined in §2 as octave band-limited fields. We wish now to generalize to bands of any constant logarithmic width $k_n/k_{n-1} = \alpha$, where k_n is the geometric-mean wavenumber of band n. For example, α could be 10. Suppose that the spatial distribution of $u_n(\mathbf{x}, t) \equiv |\mathbf{u}^n(\mathbf{x}, t)|$ in an inertial-range band is strongly intermittent, so that for most $\mathbf{x}$ a good approximation is $u_n(\mathbf{x}, t) = 0$. (In this section subscripts denote bands and are *not* tensor indices.) Let $P_n(u, t)\,du$ be the fraction of the total volume in which, on the average, u_n has a (non-negligible) value between u and $u+du$. The total probability distribution of u_n is then approximated by

$$P_n(u, t) + \left[1 - \int_0^\infty P_n(u', t)\,du'\right]\delta(u).$$

Now let the structure in band $n-1$ disintegrate into n-band structures, with a disintegration rate-constant $\beta k_{n-1} u_{n-1}$, where β is a numerical coefficient and

$k_{n-1}u_{n-1}$ is the local eddy-circulation frequency for band $n-1$. Let each breakdown yield N_n successive non-overlapping daughter structures, each with the same volume as the mother and with amplitudes satisfying the energy-conservation relation $N_n u_n^2 = u_{n-1}^2$ locally. Thus the disintegration of $(n-1)$-band structures with total probability measure $P_{n-1}(u,t)\,du$ in the interval

$$u < u_{n-1} < u+du$$

gives rise to daughters with total measure $N_n P_{n-1}(u,t)\,du$ in the interval

$$u < N_n^{\frac{1}{2}} u_n < u+du.$$

If this process goes on simultaneously in all inertial-range bands, it is described by the rate equation

$$dP_n(u)/dt = -\beta k_n u P_n(u) + \beta N_n k_{n-1}(uN_n^{\frac{1}{2}})\, P_{n-1}(uN_n^{\frac{1}{2}})\, N_n^{\frac{1}{2}}. \tag{5.1}$$

Here u is now the daughter amplitude, $\beta k_n u$ the daughter disintegration rate, $uN_n^{\frac{1}{2}}$ and $\beta k_{n-1} uN_n^{\frac{1}{2}}$ are, respectively, the mother amplitude and disintegration rate, while $P_{n-1}(uN_n^{\frac{1}{2}})\,N_n^{\frac{1}{2}}\,du$ is the probability measure of mothers that yield daughters with $u < u_n < u+du$. Integration of (5.1), multiplied by u^2, gives the energy balance equation

$$d\langle u_n^2\rangle/dt = -\beta k_n \langle u_n^3\rangle + \beta k_{n-1}\langle u_{n-1}^3\rangle. \tag{5.2}$$

The energy input term for band n is the energy loss term for band $n-1$, which exhibits the conservation built into the model. Here

$$\langle u_n^r\rangle \equiv \int_0^\infty P_n(u)\, u^r\, du.$$

The statistically steady state is found by setting $dP_n(u)/dt = 0$ in (5.1). Integration of the resulting equation, after multiplication by u^{r-1}, yields

$$k_n\langle u_n^r\rangle = k_{n-1}\langle u_{n-1}^r\rangle\, N_n^{\frac{1}{2}(3-r)}. \tag{5.3}$$

The requirement that (5.3) give a similarity solution, in which the ratios

$$\langle u_n^r\rangle/\langle u_n^s\rangle^{r/s}$$

are independent of n, now implies that $N_n = (k_n/k_{n-1})^{\frac{2}{3}}$. With this choice, (5.3) yields

$$\langle u_n^r\rangle/\langle u_{n-1}^r\rangle = (k_n/k_{n-1})^{-\frac{1}{3}r}, \tag{5.4}$$

in agreement with the 1941 Kolmogorov theory.

Now let a stochastic element be introduced into the disintegration process by again assuming that each $(n-1)$-band structure gives $N_n = (k_n/k_{n-1})^{\frac{2}{3}}$ n-band daughters but with the amplitudes of mother and daughter related by

$$u_{n-1} = u_n N_n^{\frac{1}{2}}/x,$$

where x is a positive random variable with probability distribution

$$Q(x) \quad \left[\int_0^\infty Q(x)\,dx = 1\right].$$

Then (5.1) is replaced by

$$dP_n(u)/dt = -\beta k_n u P_n(u) + \beta k_{n-1} N_n^2 u \int_0^\infty P_{n-1}(uN_n^{\frac{1}{2}}/x)\, Q(x)\, x^{-2}\,dx. \tag{5.5}$$

Here $\beta k_{n-1} u N_n^{\frac{1}{2}}/x$ is the mother disintegration rate and $P_{n-1}(uN_n^{\frac{1}{2}}/x)\, N_n x^{-1} du$ is the conditional probability measure of mothers that yield daughters with

$$u < u_n < u + du.$$

Multiplication of (5.5) by u^2, followed by integration first over u and then over x, gives the energy balance equation

$$d\langle u_n^2 \rangle / dt = -\beta k_n \langle u_n^3 \rangle + \beta k_{n-1} \langle u_{n-1}^3 \rangle \langle x^2 \rangle, \tag{5.6}$$

where $$\langle x^r \rangle \equiv \int_0^\infty Q(x)\, x^r\, dx.$$

In order that there be energy conservation in the mean, $Q(x)$ must therefore be constrained by $\langle x^2 \rangle = 1$. We interpret the non-conservation in individual disintegrations ($x \neq 1$) as reflecting the fluctuations associated with energy transport from point to point in physical space. The original model is recovered by taking $Q(x) = \delta(x-1)$.

The moment relations in the statistically steady state are obtained from (5.5) by setting $dP_n(u)/dt = 0$, multiplying by u^{r-1} and integrating over u first and then x. This gives

$$\langle u_n^r \rangle / \langle u_{n-1}^r \rangle = (k_n/k_{n-1})^{-\frac{1}{3}r} \langle x^{r-1} \rangle, \tag{5.7}$$

which may be rewritten as

$$\langle u_n^r \rangle / \langle u_{n-1}^r \rangle = (k_n/k_{n-1})^{-\frac{1}{3}r+\mu(r)}, \tag{5.8}$$

where $$\mu(r) = \ln \langle x^{r-1} \rangle / \ln \alpha. \tag{5.9}$$

Since $Q(x)$ is a probability distribution, the x moments satisfy realizability inequalities, of which examples are

$$\langle x \rangle^2 \leqslant \langle x^2 \rangle, \quad \langle x^{2r} \rangle \geqslant \langle x^2 \rangle^r, \quad \langle x^r \rangle \geqslant \langle x^s \rangle^{r/s} \quad (r > s \geqslant 1). \tag{5.10}$$

The equality signs hold only for the degenerate case of δ-function Q. Since $\langle x^2 \rangle = 1$, it follows that

$$\mu(2) < 0, \quad \mu(3) = 0, \quad \mu(r) > 0 \quad (r > 3). \tag{5.11}$$

A consistency requirement is

$$\int_0^\infty P_n(u)\, du < 1.$$

This can hold as $n \to \infty$ only if $\mu(0) < 0$, or $x^{-1} < 1$. Any $Q(x)$ of the form

$$Q(x) = f(x, x^{-1})\, x^{-1}, \quad f(x,y) = f(y,x), \tag{5.12}$$

gives $\langle x^{-1} \rangle = \langle x \rangle$, and so automatically satisfies this requirement.

The power-law behaviour defined by (5.8) and (5.11) is qualitatively in agreement with the 1962 Kolmogorov theory. The band energy $\langle u_n^2 \rangle$ falls off more rapidly than $k_n^{-\frac{2}{3}}$, while normalized high-order moments grow with order and with n, indicating increasing intermittency as n increases. The flexibility to fit a wide range of experimental data is provided by the freedom of choice of $Q(x)$ and α in the model. The physics of the model is more plausible if $Q(x)$ falls to zero for x greater than some cut-off (say $x = 2$) than if $Q(x)$ extends to infinite x. Cut-off forms of $Q(x)$ also turn out to be good choices for matching the experimental values $\mu(2) \sim -0{\cdot}05$, with α in the range 2–4, although it is not our purpose here to make detailed matches with experiment.

The agreement with the 1962 theory does not extend to log-normality. First of all, the moment ratios (5.8) depend on the precise statistics of x, even as $n \rightarrow \infty$ (again cf. Novikov 1971). Second, the asymptotic distribution to which $P_n(u)$ tends as $n \rightarrow \infty$ is not log-normal; this is because $\langle x^{r-1} \rangle$ rather than $\langle x^r \rangle$ appears in (5.7). If the exponent were r, equation (5.7) could be realized by random variables obeying

$$u_n = (k_n/k_{n-1})^{-\frac{1}{3}} x_n u_{n-1}, \tag{5.13}$$

where the x_n are statistically independent variables with probability distribution $Q(x)$. Then it would follow that u_n is asymptotically log-normal.

The failure of asymptotic log-normality in the present model is associated with nonlinearity in the disintegration process: the rate $\beta k_n u_n$ is not a constant but instead depends on u_n. Consider what happens if $\beta k_n u_n$ is replaced by the typical eddy frequency $(\epsilon k_n^2)^{\frac{1}{3}}$, which is suggested by the 1941 Kolmogorov theory. The corresponding changes in (5.5) are that $\beta k_n u$ in the first term on the right-hand side is replaced by $(\epsilon k_n^2)^{\frac{1}{3}}$ while a factor $\beta k_{n-1} u N_n^{\frac{1}{3}}/x$ extracted from the second term is replaced by $(\epsilon k_{n-1}^2)^{\frac{1}{3}}$. The easily verified result is that (5.7) is changed to

$$\langle u_n^r \rangle / \langle u_{n-1}^r \rangle = (k_n/k_{n-1})^{-\frac{1}{3}r} \langle x^r \rangle. \tag{5.14}$$

The qualitative behaviour of the model is essentially unchanged, with regard to power laws, but now the u_n distribution *is* asymptotically log-normal as $n \rightarrow \infty$. It seems more natural, within the general context of the model, to use the local eddy frequency $k_n u_n$ rather than a frequency determined by the mean dissipation ϵ.

The nonlinearity reflected in (5.7) is associated with a disintegration rate proportional to amplitude, which tends to depress the relative probability of occurrence of large amplitudes. Thus, for given $Q(x)$, equation (5.7) gives a less rapid rise of, for example, kurtosis with n than does (5.14). It is interesting that the effect of nonlinearity in the present conservative steady-state model is opposite in sense to that noted previously for the non-conservative transient problem posed by (4.1).

In § 2, we invoked spatial diffusion within the band-limited fields as a mechanism of cross-linking cascade chains and thereby possibly validating the 1941 theory. The diffusion process, like the band-to band cascade, presumably should have a characteristic rate of order $k_n u_n$. The simplest way to represent such a process in the present model would appear to be the inclusion of an additional disintegration term of the form $-\beta' k_n P_n(u)$ in (5.5), plus a creation term describing the reappearance of the excitation in the same band n, but with a different amplitude distribution. The diffusion must be conservative, which means that the contributions of the additional destruction and creation terms to the energy balance equation must cancel. In turn, this implies that the right-hand side of (5.6) must still vanish in the statistically steady state; in other words, (5.7) is unaltered for $r = 3$. An important conclusion can now be drawn without specifying the precise form of the creation term except to note that it must provide a positive contribution to each steady-state moment equation formed from (5.5). Noting this, and taking account of the extra destruction term, one now finds

$$\langle u_n^r \rangle / \langle u_{n-1}^r \rangle \geqslant [\beta/(\beta+\beta')]\,(k_n/k_{n-1})^{-\frac{1}{3}r} \langle x^{r-1} \rangle \tag{5.15}$$

instead of (5.7). This equation, together with the unaltered equation (5.7) for $r = 3$, implies that

$$\langle u_n^r \rangle / \langle u_n^3 \rangle^{\frac{1}{3}r} \geqslant \langle u_{n-1}^r \rangle / \langle u_{n-1}^3 \rangle^{\frac{1}{3}r} [\beta/(\beta+\beta')] \langle x^{r-1} \rangle, \tag{5.16}$$

where we have used $\langle x^2 \rangle = 1$.

If $Q(x)$ satisfies $\langle x^2 \rangle = 1$, and has support away from $x = 1$, then $\langle x^r \rangle$ must grow at least exponentially with r, for large r. Therefore (5.16) shows that normalized moments at large enough r must grow with n, no matter how large the diffusion parameter β' may be. The conclusion is that, in the present model at least, the cross-linking of cascade chains by intra-band mixing cannot be powerful enough to suppress the building up of intermittency along the cascade chains. The fact that intense eddies break up faster does not suffice to limit intermittency.

6. Kolmogorov's first hypothesis

We wish now to discuss possible alternatives to Kolmogorov's first hypothesis of 1962. In order to bring out the non-uniqueness of this hypothesis, it will be sufficient, and clarifying, to restrict attention to statistically stationary and homogeneous turbulence, maintained in an infinite cyclic box by stationary homogeneous stirring forces that are confined to low wavenumbers. We shall suppose, specifically, that a random force $f_i^n(\mathbf{x}, t)$ is added to the right-hand side of (2.5) and that it vanishes except for $n = 0$. We shall use the band-limited fields $\mathbf{u}^n(\mathbf{x}, t)$ as a framework, rather than the velocity differences

$$\mathbf{u}(\mathbf{x}+\mathbf{r}, t) - \mathbf{u}(\mathbf{x}, t),$$

because the former fit more naturally into discussions of energy transfer. The field $\mathbf{u}^n(\mathbf{x}, t)$ may be taken in the generalized form used in §5. That is, $\mathbf{u}^0(\mathbf{x}, t)$ comprises all wavenumbers less than k_0 and $\mathbf{u}^n(\mathbf{x}, t)$ for $n > 0$ comprises all wavenumbers satisfying $\alpha^{n-1} < k/k_0 < \alpha^n$, while the characteristic wavenumbers k_n $(n > 0)$ are defined by $k_n = \alpha^{n-\frac{1}{2}} k_0$. Then k_0 is the macroscale wavenumber and α may be 2, or 10, or any bandwidth which seems appropriate.

We shall restrict $\mathbf{f}^0(\mathbf{x}, t)$ so as to exclude intense spatial or temporal intermittency in the forcing which, following the discussion of §2, could make the 1941 theory *a priori* inapplicable. This can be accomplished, for example, by requiring that all but the first few terms of an expansion of $\mathbf{f}^0(\mathbf{x}, t)$ in Wiener–Hermite functionals of a space–time white-noise process vanish. Here we appeal to the four-dimensional generalization of the Wiener–Hermite expansion discussed by Imamura, Meecham & Siegel (1965) and others.

The two hypotheses of the 1941 theory may now be formulated as follows.

First hypothesis. At very high Reynolds numbers, the multivariate distributions of any set of the $\mathbf{u}^n(\mathbf{x}, t)$ $(n \gg 1)$ at a given instant are universal functions solely of the k_n in the set and of ϵ and ν.

Second hypothesis. As the Reynolds number becomes infinite, these distributions become independent of ν, for fixed n values $\gg 1$.

The question now is what modifications of the first hypothesis are plausible, and consistent with the concept of a multistage cascade proceeding in constant logarithmic steps of wavenumber, and with intermittency increasing in some systematic and self-similar fashion at each step. Clearly the asymptotic distributions must now depend on the number of cascade steps, which is

$$\sim \ln (k_n/k_0) \sim n.$$

Therefore the minimum modification of the first hypothesis that embodies the qualitative ideas underlying the 1962 theory would appear to be the following: at very high Reynolds numbers the multivariate distributions of any set of the $\mathbf{u}^n(\mathbf{x},t)$ $(n \gg 1)$ at a given instant are universal functions solely of the k_n in the set, of k_0 and of ϵ and ν. No modification of the second hypothesis seems to be required. In order to take account of possible persistence of intermittency introduced through the $\mathbf{f}^0$ distribution, the first hypothesis could be weakened by giving to the word universal the special meaning that logarithms of probability densities or moments become universal functions, thereby ignoring factors of order one.

Even without this weakening qualification, the modified hypothesis we have stated does not permit specific conclusions about functional dependence in the inertial range, analogous to those of the 1941 theory. For example, it implies for $E(k)$ only that

$$E(k) = \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} g(k/k_0), \tag{6.1}$$

where g is some unknown universal function. There is no basis, in the hypothesis itself, for concluding that $g(k/k_0)$ must be a power rather than, say, a power multiplied by some logarithmic function of k/k_0. The latter conceivably could arise from some subtle phenomenon of alignment of vortex lines in the straining field as the cascade proceeds.

In his 1962 paper, Kolmogorov makes a stronger first hypothesis than we do above, by making detailed assumptions about conditional probabilities. In fact, the paper contains two, inequivalent, versions of the first hypothesis. They go as follows for the ν-independent inertial range. The first version states that, if $\mathbf{x}'$ is in the neighbourhood of $\mathbf{x}$, then the probability distribution of

$$\mathbf{u}(\mathbf{x}',t) - \mathbf{u}(\mathbf{x},t)$$

conditional upon a sharp value of $\tilde{\epsilon}_r(\mathbf{x},t)$ is universal, provided that $|\mathbf{x}'-\mathbf{x}|$ and r are both in the inertial range and that lengths and velocities are non-dimensionalized with r and $\tilde{\epsilon}_r(\mathbf{x},t)$. The second version states that the probability distribution of the ratio

$$[u_i(\mathbf{x}^{(n)},t) - u_i(\mathbf{x},t)]/[u_i(\mathbf{x}^{(0)},t) - u_i(\mathbf{x},t)]$$

is a universal function of the arguments $(\mathbf{x}^{(n)}-\mathbf{x})/|\mathbf{x}^{(0)}-\mathbf{x}|$, where the

$$\mathbf{x}^{(n)} - \mathbf{x} \quad (n = 0, 1, 2 \ldots)$$

are any set of vectors which all lie in the inertial range.

The two versions are inequivalent because, taking $\mathbf{x}^{(0)} = \mathbf{x}+\mathbf{r}$, a sharp value of $\tilde{\epsilon}_r(\mathbf{x}, t)$ corresponds, in general, to a statistical spread of values of

$$u_i(\mathbf{x}^{(0)}, t) - u_i(\mathbf{x}, t),$$

and vice versa. Predictions for the probability distribution of $\mathbf{u}(\mathbf{x}', t) - \mathbf{u}(\mathbf{x}, t)$ in the two versions could be equivalent, therefore, only if the normalized joint distribution of $\tilde{\epsilon}_r(\mathbf{x}, t)$ and $u_i(\mathbf{x}+\mathbf{r}, t) - u_i(\mathbf{x}, t)$ were universal. But the latter situation would be inconsistent with the build-up of intermittency and change of statistics in general during the cascade; it would make sense only in the degenerate case of the 1941 theory.

A further difference in the two versions is that $\mathbf{u}(\mathbf{x}^{(0)}, t) - \mathbf{u}(\mathbf{x}, t)$ is an inertial-range quantity while $\tilde{\epsilon}_r(\mathbf{x}, t)$ is, instead, the integral of a dissipation-range quantity and cannot be directly constructed as a function of inertial-range quantities. Consequently, the second version leads directly to the prediction of power-law dependence in the inertial range, while the first version does not, unless augmented with some additional assumption, like Kolmogorov's third (log-normal) hypothesis. We shall demonstrate the power-law implication shortly. First we wish to expand on the lack of a direct connexion between $\tilde{\epsilon}_r(\mathbf{x}, t)$ and inertial-range quantities. The overall dissipation rate ϵ must equal the rate of energy cascade through the inertial range, in the statistically steady state. However, conservation alone does not give any corresponding relation between $\tilde{\epsilon}_r(\mathbf{x}, t)$ and the local inertial-range cascade rate as measured, say, by $\Pi^n(\mathbf{x}, t)$ defined in (2.6). *A priori*, $\tilde{\epsilon}_r(\mathbf{x}, t)$ and

$$\Pi^n_r(\mathbf{x}, t) = \int_{|\mathbf{y}|<r} \Pi^n(\mathbf{x}+\mathbf{y}, t)\, d^3\mathbf{y}$$

need not be similar even in their typical order of magnitude. For example, it could be that such a strong build-up of *temporal* intermittency occurs in the inertial cascade that $\tilde{\epsilon}_r(\mathbf{x}, t)$ is strongly intermittent in time and only its time integral over the eddy time $(k_n u_n)^{-1}$ is comparable in typical magnitude with $\Pi^n_r(\mathbf{x}, t)$, in the limit of very high Reynolds number.

Thus, if a hypothesis is desired that asserts universal probabilities conditional on sharp values of a cascade parameter of local significance, something like $\Pi^n(\mathbf{x}, t)$ would seem an appropriate basis for constructing the parameter, rather than $\tilde{\epsilon}(\mathbf{x}, t)$. However, there seems to be an infinite number of inequivalent candidates for conditional-probability hypotheses, and little way to choose among them without knowing more than is known about the mechanics of the Navier–Stokes equation. For example, consider the hypothesis that the distribution of the band-limited fields is such that $|\mathbf{u}^n(\mathbf{x}, t)|/|\mathbf{u}^m(\mathbf{x}, t)|$ has a distribution which is a universal function of k_n/k_m. This can be expressed in the conditional-probability form

$$P_n(u) = \int P_{nm}(u|u')\, P_m(u')\, du', \tag{6.2}$$

where $P_{nm}(u|u')$ has the form

$$P_{nm}(u|u') = F(u/u', k_n/k_m)/u' \tag{6.3}$$

and $P_n(|\mathbf{u}^n(\mathbf{x},t)|)$ is the probability distribution of $|\mathbf{u}^n(\mathbf{x},t)|$. Manipulations like those of § 5 then readily show that

$$\langle u_n^r \rangle = \langle u_m^r \rangle f_r(k_n/k_m), \tag{6.4}$$

where f_r is a universal function and, as in § 5, $u_n = |\mathbf{u}^n(\mathbf{x},t)|$. Now, if (6.4) is applied between bands n and m, then between bands m and s, and finally between bands n and s, the result is

$$f_r(k_n/k_m) f_r(k_m/k_s) = f_r(k_n/k_s), \tag{6.5}$$

whence f_r is a power of its argument. A similar demonstration shows that Kolmogorov's second version of the first hypothesis yields power laws, as stated above.

Now, however, consider, instead of (6.2), the equally plausible hypothesis

$$\begin{aligned} P_n(u) &= \iint P_{nm}(u|v,w)\, P_m(v)\, P'_m(w)\, dv\, dw, \\ P'_n(u) &= \iint P'_{nm}(u|v,w)\, P_m(v)\, P'_m(w)\, dv\, dw, \end{aligned} \tag{6.6}$$

where P_n is again the distribution of u_n, P'_n is the distribution of

$$|k_n^{-1} \sum_{ij} \partial u_i^n(\mathbf{x},t)/\partial x_j|,$$

and P_{nm} and P'_{nm} are universal functions of k_n/k_m, u/v and u/w. Such a conditional-probability hypothesis is a simple expression of an anticipation that the magnitude of velocity derivatives as well as velocity amplitudes may be important in determining the instabilities that lead to breakdown. (It might be just as plausible to include second derivatives as well.) Because there are now the two arguments u/v and u/w, the demonstration of a power law for moments does not go through and the hypothesis does not lead to definite inertial-range predictions.

All the hypotheses considered in this section so far are relations between probabilities all measured at the same instant. This in itself seems an inappropriate limitation, since the Navier–Stokes equation gives evolution in time, and the inertial range is a state of statistical non-equilibrium, in the fundamental sense. The model of § 5, which used a rate equation for partial distributions, represents a slight relaxation of this limitation. Already that relaxation was sufficient to give a non-log-normal asymptotic distribution, while the related instantaneous model, described by (6.2)–(6.4), is readily seen to give log-normal asymptotic statistics. A proper treatment of many-time distributions in the inertial range probably requires the introduction of some kind of Lagrangian or modified Lagrangian description (Kraichnan 1964).

The principal point of this section, and § 5, is that the assumption that there exists a self-similar cascade mechanism, local in scale size, is not in itself sufficient to determine the inertial-range statistics, even to the extent of implying a power law for the energy spectrum. Although a wide variety of choices of mechanism involving conditional-probability assumptions for the cascade will lead to increases of intermittency with decreasing scale size, only some of these choices will lead to log-normal asymptotic statistics.

The situation with regard to dissipation-range statistics is even less well defined. We have pointed out above that there is no very direct relation between inertial-range and dissipation-range statistics. Moreover, strong intermittency in the far-dissipation range, which means strong intermittency of high-order velocity derivatives, is to be expected at all Reynolds numbers, high or low, independently of whether there is an inertial-range cascade (Kraichnan 1967). This is simply a consequence of the fact that when the spectrum falls off very rapidly with wavenumber, even slight fluctuations in spectrum parameters from region to region in the fluid produce enormous changes in these derivatives.

As an example, suppose that the local energy spectrum at high wavenumbers has the form

$$E(k, \mathbf{x}) \propto \exp[-k/a(\mathbf{x})]/a(\mathbf{x}), \tag{6.7}$$

where $a(\mathbf{x})$ is a positive random variable that changes slowly with $\mathbf{x}$. Then local averages of velocity derivatives have the form

$$\langle(\partial^n u/\partial x^n)^2\rangle_x \propto [a(\mathbf{x})]^{2n}. \tag{6.8}$$

For large n, $[a(\mathbf{x})]^{2n}$ is a highly intermittent function of $\mathbf{x}$ even when $a(\mathbf{x})$ itself is only mildly intermittent.

7. Statistical mechanics of the cascade

The inertial-range cascade is an extremely interesting process from the viewpoint of fundamental statistical mechanics. Suppose that $\nu = 0$ and that the Navier–Stokes equation is truncated by removing all terms involving wavenumbers above some cut-off K. This system obeys Liouville's theorem, and there is an absolute equilibrium ensemble in which the Fourier components are statistically independent, Gaussianly distributed and in equipartition (Lee 1952). Now suppose that an initial distribution is set up in which the energy is contained wholly in a band of wavenumbers $k < k_0 \ll K$ and in which the excited wavenumbers have Gaussian amplitude distributions. Presumably the system will evolve eventually into the Gaussian equipartition solution, provided that the initial ensemble has zero mean momentum, angular momentum and helicity (Kraichnan 1973). However, if the general ideas of Kolmogorov's (1962) theory are correct, the evolution from one Gaussian state to the other will involve a transient period in which the inertial-range cascade operates and the higher degrees of freedom have highly intermittent distributions in $\mathbf{x}$ space. The question arises: is such behaviour characteristic in general of the approach to equilibrium of nonlinear systems with many degrees of freedom? If so, are there any general statistical-mechanical principles which govern the non-Gaussian transient stage?

The behaviour described is, of course, only conjectural for the Navier–Stokes equation. However, it can be demonstrated for some model systems of the form

$$dy_n/dt = \sum_m a_{nm} y_m \quad (a_{nm} = -a_{mn}), \tag{7.1}$$

where the a_{nm} are random coefficients, suitably chosen to give chain-type multifurcating couplings.

If the Navier–Stokes equation is not truncated, and one takes the limit $\nu \rightarrow 0$, then the 1962 theory implies that the dissipation takes place in an infinitesimal fraction of the total fluid volume. Again this is conjectural for the Navier–Stokes equation, but it is demonstrable for Burgers' equation, which has the same kind of nonlinearity and invariances and is known to lead to shock waves. Again, the question arises of how general and significant this behaviour may be.

We wish to argue now that, even if the inertial-range cascade with increasing intermittency reflects some general statistical-mechanical principle, the precise forms of the inertial-range statistical functions and, in particular, the value of μ in the modified spectrum law (1.9) cannot follow from general considerations but depend instead on the detailed structure of the Navier–Stokes equation. A contrary view has been discussed by Nelkin (1973), who draws attention to a possible analogy between μ and the universal exponents in critical-point phenomena.

Our basic point is that the inertial-range cascade represents strong statistical disequilibrium. This carries two implications. First, that analogies with equilibrium and near-equilibrium phenomena are unjustified. Second, that the structure of the inertial range depends on the actual magnitude of the coefficients coupling the degrees of freedom and not just on their overall symmetry and invariance properties. This is because the cascade is a transport process and the coefficient magnitudes affect the rate of transport.

We can support this point of view by considering the generalized Navier–Stokes equation

$$(\partial/\partial t - \nu\nabla^2)\, u_i(\mathbf{x}, t) = -P_{ij}(\nabla)\,[\mathbf{v}(\mathbf{x}, t) . \nabla u_j(\mathbf{x}, t)], \tag{7.2}$$

where $P_{ij}(\nabla)$ is the solenoidal projection operator and $\mathbf{v}(\mathbf{x}, t)$ is a solenoidal functional of $\mathbf{u}(\mathbf{x}, t)$ that satisfies

$$\int \mathbf{v}(\mathbf{x}, t)\, d^3\mathbf{x} = \int \mathbf{u}(\mathbf{x}, t)\, d^3\mathbf{x}.$$

With these conditions, (7.2) gives conservation of $\int |\mathbf{u}(\mathbf{x}, t)|^2\, d^3\mathbf{x}$ by the right-hand side, exhibits Galilean invariance (Kraichnan 1964) and has the same inviscid equilibrium equipartition distribution as the Navier–Stokes equation. The latter is recovered by putting $\mathbf{v}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t)$.

The inertial-range cascade properties clearly depend on what $\mathbf{v}(\mathbf{x}, t)$ is. For example, consider

$$\mathbf{v}(\mathbf{x}, t) = \exp\,[\lambda^2\nabla^2]\,\mathbf{u}(\mathbf{x}, t), \tag{7.3}$$

where λ is an intrinsic length. This makes $\mathbf{v}(\mathbf{x}, t)$ a non-local functional, but $P_{ij}(\nabla)$, which is in the original Navier–Stokes equation, is already more non-localizing. Suppose that a statistically steady state is maintained by driving the system at wavenumbers below λ^{-1} with a forcing term $f_i(\mathbf{x}, t)$ on the right-hand side. Equation (7.3) shows that the effective shear field acting on scales very much less than λ is confined to the input wavenumbers. Consequently, although the energy cascade is still local in wavenumber, the entire basis of the $-\frac{5}{3}$- or $(-\frac{5}{3} - \mu)$-type inertial range is gone. Instead, the arguments of Batchelor

(1959) apply, and one is led to infer that the inertial-range spectrum has the form

$$E(k) \sim (\epsilon/v_0)\,\lambda k^{-1}, \tag{7.4}$$

where v_0 is characteristic of the velocity in scales less than λ. This complete difference in non-equilibrium behaviour arises despite the fact that the modified equation has the same essential invariances, symmetries, dimensionality and equilibrium statistical ensembles as the Navier–Stokes equation.

With regard to the Navier–Stokes equation itself, the value of μ in (1.9), if that equation is valid, depends basically on how fast intermittency rises along the cascade chain. This in turn depends on what the effective step size is in wave-number, and what the effective statistical spread (analogous to the function Q in the model of § 5) is at each step. Neither of these factors would appear to have any relation to basic invariances or symmetries; instead they would appear to be complicated functions of the particular mode-coupling coefficients that are present in the Navier–Stokes equation.

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